# Graphic Solution of 2xn and mx2 Games.

Graphical method is applicable to only those games in which one of the player has only two strategies. This method reduces the 2xn or mx2 game to 2x2 size by identifying and then eliminating the dominant strategies and then solve it by the analytical (algebric or arithmatic) methods. The resulting solution is the solution to the original problem. This method explain the idea of saddle point graphically.

Consider the following 2xn games:-

$$\begin{array}{ccccc} B \rightarrow & y_1 & y_2 & \cdots & y_n \end{array}$$

$$\text{Player A} \downarrow \quad \begin{matrix} x_1 \\ 1-x_1 \end{matrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$$

It's assumed that game has no saddle point. Player A has two strategies $A_1$ and $A_2$ which he has strategy of playing with prob. $x_1$ & $1-x_1$ where $0 \leq x_1 \leq 1$

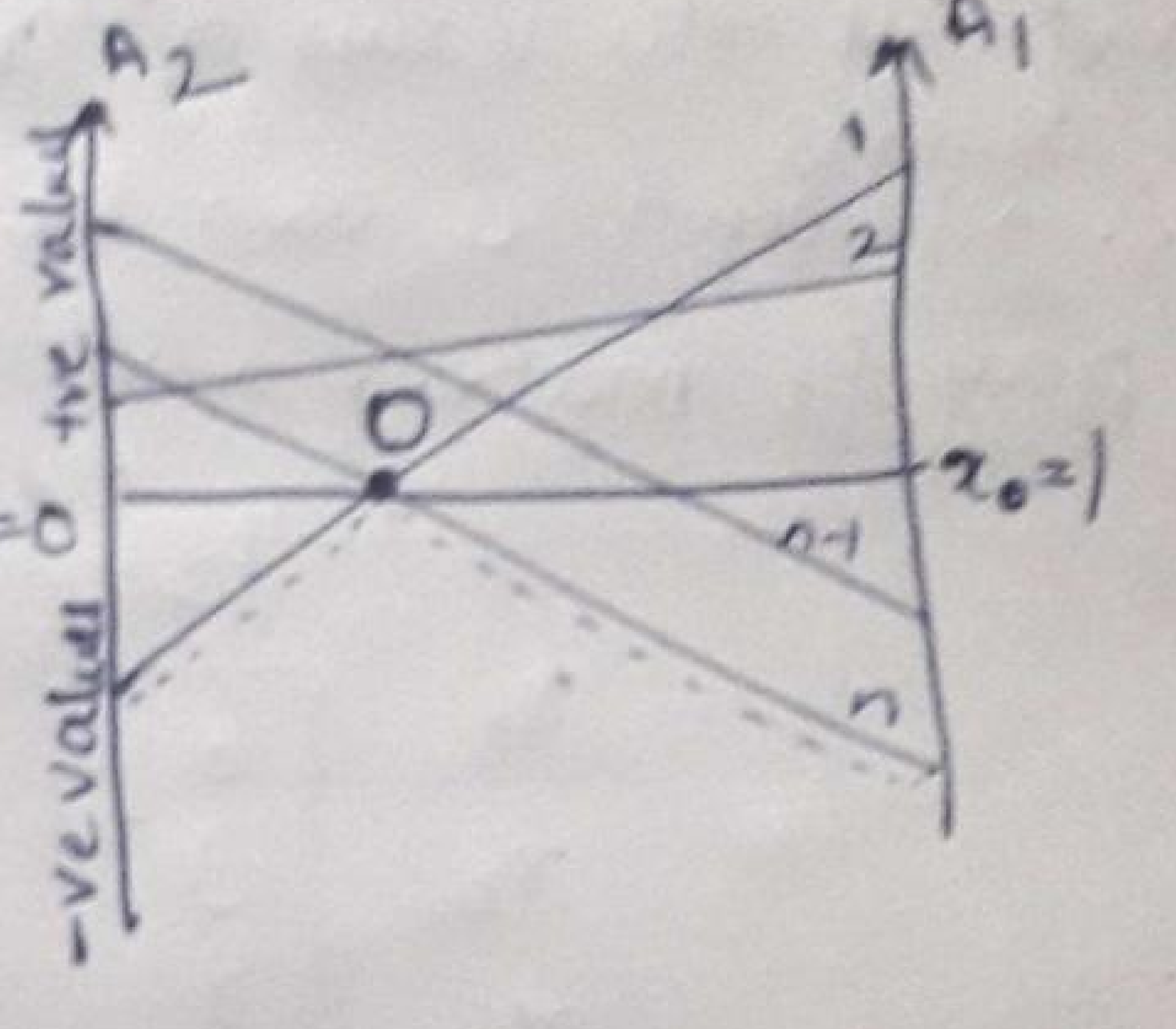
Player B has n strategies which he mixes with probabilities $y_1, y_2 \ldots y_n$ where $y_i \geq 0 \ \forall i = 1 \ldots n$ & $\sum_{i=1}^{n} y_i = 1$

Obj is to determine optimal value of $x_1$

Expected Payoff for A corresponding to the pure strategies of B are given below:

| B's pure strategy | A's expected payoff |
|---|---|
| 1 | $a_{11}x_0 + a_{21}(1-x_0) = (a_{11}-a_{21})x + a_{21}$ |
| 2 | $a_{12}x + a_{22}(1-x) = (a_{12}-a_{22})x + a_{22}$ |
| $\vdots$ | |
| n | $a_{1n}x + a_{2n}(1-x) = (a_{1n}-a_{2n})x + a_{2n}$ |

∴ A's expected payoffs vary linearly with $x$. Now according to the maximin criterion for mixed strategies, A should select the value of $x$ which maximizes his minimum expected payoff. This may be done by plotting the above lines as a function of $x$. As shown graphically.




If A chooses strategy $A_1$ then $x_1=1$ and prob with which he chooses strategy $A_2$ i.e $x_2=0$

Prob $x_1$ is represented by the horizontal lines and strategies $A_2$ and $A_1$ are represented by two vertical lines unit distance apart.

A's expected payoff are represented by the slopping lines. A number is alloted to each line corresponding to B's pure strategy. The lower boundry (enevelope) of these lines (shown by ---- lines) gives the minimum expected payoff to A as a function of $x$. A is a maximising player. the heighest point on this lower boundary (O) gives the maximin expected payoff to A hence the optimum value of $x_1$

Example:- Solve the given game by graphical method

$$A \quad \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ 19 & 6 & 7 & 5 \\ 7 & 3 & 14 & 6 \\ 12 & 8 & 18 & 4 \\ 8 & 7 & 13 & -1 \end{bmatrix} \quad B$$

Solⁿ first look for the saddle point using maximin minimax method, which does not exist in the problem.

Now reduce the game by dominance

All the values of column 1 dominate coloumn 4 & 2 and All the value of column 3 dominate the " " columns

∴ 1 and 3 can be removed from the given pay off matrix and matrix will reduces to (∵ B is minimizing player)

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{bmatrix} y_2 & y_4 \\ 6 & 5 \\ 3 & 6 \\ 8 & 4 \\ 7 & -1 \end{bmatrix}$$

In this matrix row 3 dominates row 4 ∴ row 4 can be removed and reduced matrix is

$$A \quad \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{bmatrix} y_2 & y_4 \\ 6 & 5 \\ 3 & 6 \\ 8 & 4 \end{bmatrix} \quad B$$

Let $A_1, A_2, A_3$ be the strategies which A mixes with probability $x_1, x_2$ & $x_3$ and B mixes strategy $B_2$ & $B_4$ with probability $y_2$ & $y_4 = 1$

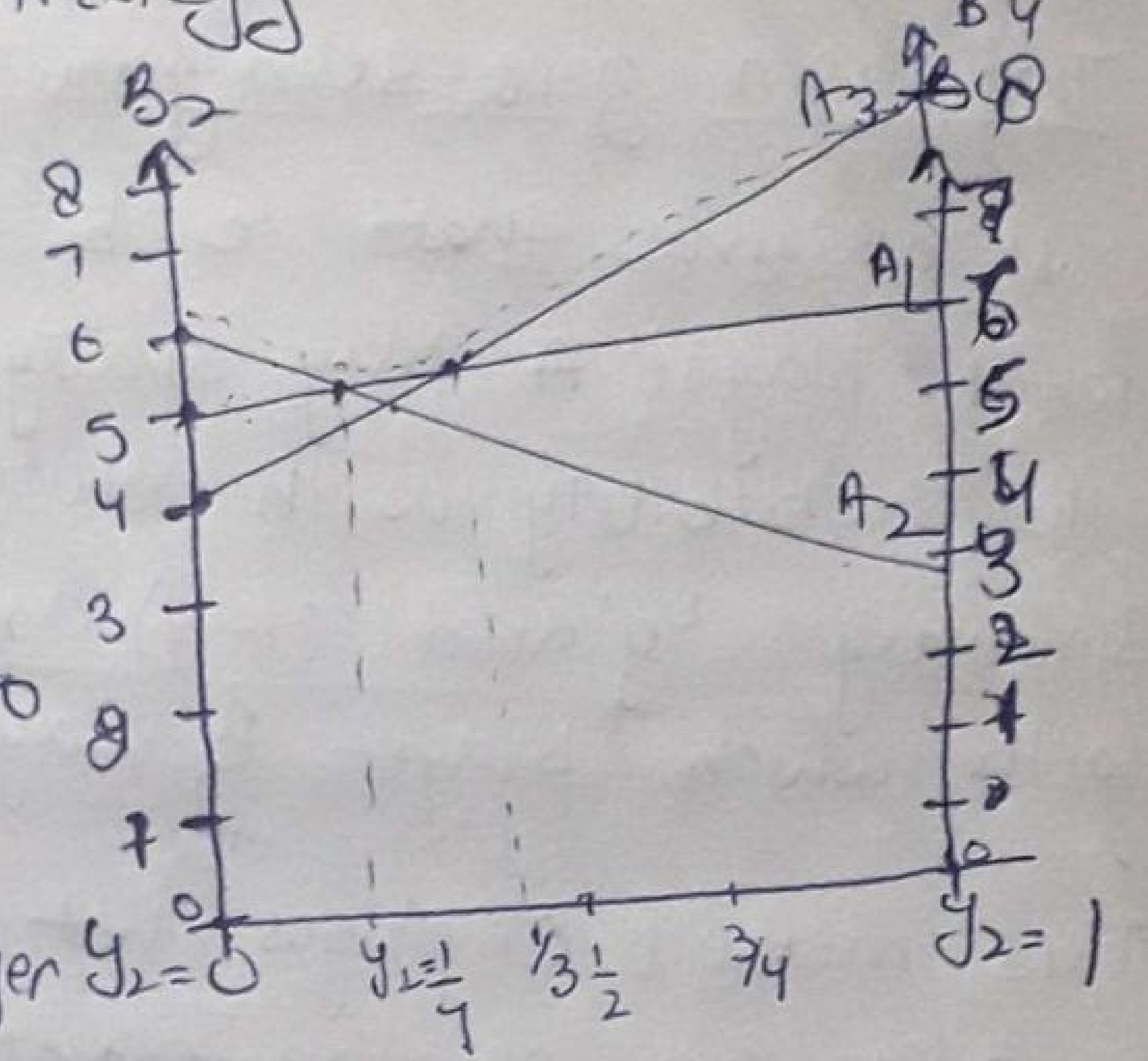
If $B_2$ adopts $B_2$ $y_2 = 1$ & $y_4 = 0$.

Now we use graphical method to solve the game. B's expected payoff corrᵈ to A's pure strategy are given below

| | | |
|---|---|---|
| $A_1$ | $6y_2 + 5(1-y_2)$ | $= y_2 + 5$ |
| $A_2$ | $3y_2 + 6(1-y_2)$ | $= -3y_2 + 6$ |
| $A_3$ | $8y_2 + 4(1-y_2)$ | $= 4y_2 + 4$ |

The three lines are straight line & are func of $y_2$
draw $B_2$ & $B_4$ parallel to each other and one unit apart and mark a scale on each of them.
To represent's A's first strategy corrosponds to $y_2 + 5$

when $y_2 = 0$ mark 5 on $B_2$
when $y_2 = 1$ mark 6 on $B_4$
and join the two points
and similarly plot other two strategies also




since B is a minimizing player bound the figure from above
as shown select the ~~min~~ lowest point from the selected bound which is the intersection point of strategy $A_1$ & $A_2$ ∴ the above 3x2 game reduces to 2x2 game which can be easily solved by arithmatic method

here $y_2 = \frac{1}{4}$

| | $y_2$ | $y_4$ | | |
|---|---|---|---|---|
| $x_1$ | 6 | 5 | 3 | 3/4 |
| $x_2$ | 3 | 6 | 1 | 1/4 |
| | 1 | 3 | | |
| | $\frac{1}{4}$ | $\frac{3}{4}$ | | |

∴ optimal strategy are
$A\left(\frac{3}{4}, \frac{1}{4}, 0, 0\right)$ $B = \left(0, \frac{1}{4}, 0, \frac{3}{4}\right)$

& $V = ~~6\times1+~~ \; 6\times\frac{3}{4}\times\frac{1}{4} + 5\times\frac{3}{4}\times\frac{3}{4} + 3\times\frac{1}{4}\times\frac{1}{4} + 6\times\frac{1}{4}\times\frac{3}{4}$

$= \frac{18}{16} + \frac{45}{16} + \frac{3}{16} + \frac{18}{16} = \frac{84}{16} = \frac{21}{4} = 5\frac{1}{4}$

*Solve the following 2 × 5 game by graphical method:*

**TABLE 9.132**

| | | | Player B | | | | |
|---|---|---|---|---|---|---|---|
| | | | 1 | 2 | 3 | 4 | 5 |
| Player A | $x_1$ | 1 | –5 | 5 | 0 | –1 | 8 |
| | $x_2 = 1 - x_1$ | 2 | 8 | –4 | –1 | 6 | –5 |

[P.U.B. Com. 2006; B.B.A

The first step is to look for a saddle point. It does not exist in the present problem. The second step is to see if the game can be reduced by dominance. In the present problem, the matrix cannot be reduced by dominance. So, let us solve the matrix by graphical method. A's expected payoffs corresponding to B's pure strategies are

| *B's pure strategies* | *A's expected payoffs* |
|---|---|
| 1 | $-5x_1 + 8(1 - x_1) = -13x_1 + 8$ |
| 2 | $5x_1 - 4(1 - x_1) = 9x_1 - 4$ |
| 3 | $0x_1 - 1(1 - x_1) = x_1 - 1$ |
| 4 | $-1x_1 + 6(1 - x_1) = -7x_1 + 6$ |
| 5 | $8x_1 - 5(1 - x_1) = 13x_1 - 5$ |

These five lines can be plotted as functions of $x_1$ as follows:

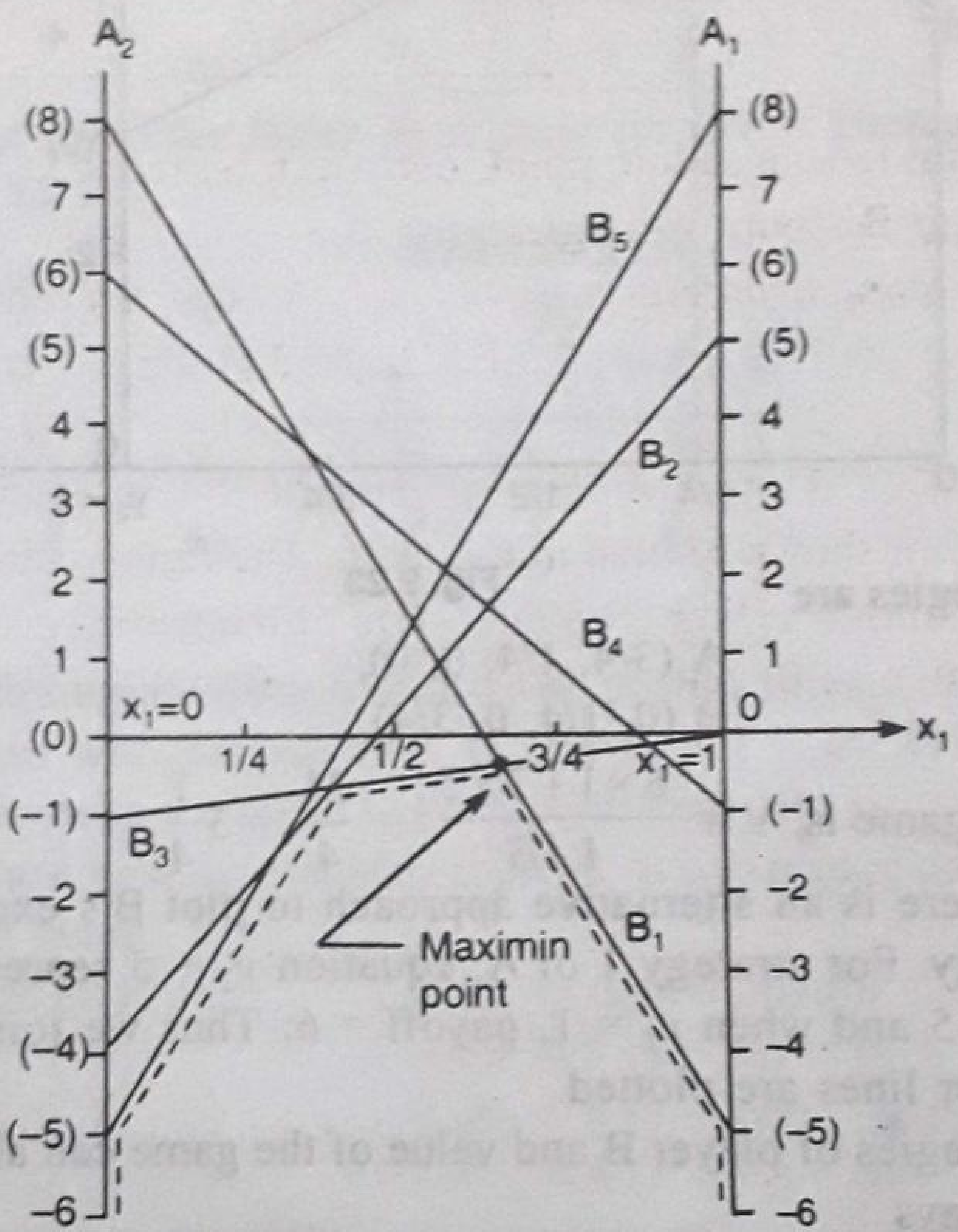



**Fig. 9.24**

Draw two parallel lines $A_1$ and $A_2$ one unit apart and mark a scale on each of them (Fig. 9.24). To represent B's first strategy, join mark $-5$ on $A_1$ with mark 8 on $A_2$; to represent B's second strategy, join mark 5 on $A_1$ with mark $-4$ on $A_2$ and so on (for the remaining three strategies) and bound the figure from below.

Since player A wishes to maximize his minimum expected payoff, the two lines which intersect at the highest point of the lower bound show the two courses of action B should choose in his best strategy, which are $B_1$ and $B_3$. We can thus immediately reduce the $2 \times 5$ game to $2 \times 2$ game which can be easily solved, say, by arithmetic method. The resulting $2 \times 2$ game is shown in table 9.133.

**TABLE 9.133**

| | | *B* 1 | 3 | | |
|---|---|---|---|---|---|
| *A* | 1 | –5 | 0 | 9 | 9/14 |
| | 2 | 8 | –1 | 5 | 5/14 |
| | | 1 | 13 | | |
| | | 1/14 | 13/14 | | |

∴ Optimum strategies are

A (9/14, 5/14), B (1/14, 0, 13/14, 0, 0),

value of the game is, $V = \frac{-5 \times 1 + 0 \times 13}{1 + 13} = \frac{-5}{14}$

**EXAMPLE 9.19-3.3**

*Solve the following game:*

**TABLE 9.134**

| | | B | | |
|---|---|---|---|---|
| | | $y_1$ | $y_2$ | $y_3$ |
| A | $x_1$ | 6 | 4 | 3 |
| | $x_2 = 1 - x_1$ | 2 | 4 | 8 |

**Solution**

This game does not have a saddle point, nor it can be reduced by dominance. A's expected payoffs corresponding to B's pure strategies are

| *B's pure strategies* | *A's expected payoffs* |
|---|---|
| 1 | $6x_1 + 2(1 - x_1) = 4x_1 + 2$ |
| 2 | $4x_1 + 4(1 - x_1) = 4$ |
| 3 | $3x_1 + 8(1 - x_1) = -5x_1 + 8$ |

These three lines are shown plotted in Fig. 9.25. The figure is bound from below. There are two maximum points C and D giving the same value of the game.

(*i*) *Point C* : Strategies $B_1$ and $B_2$ are chosen by B and the following 2 × 2 game is to be solved :

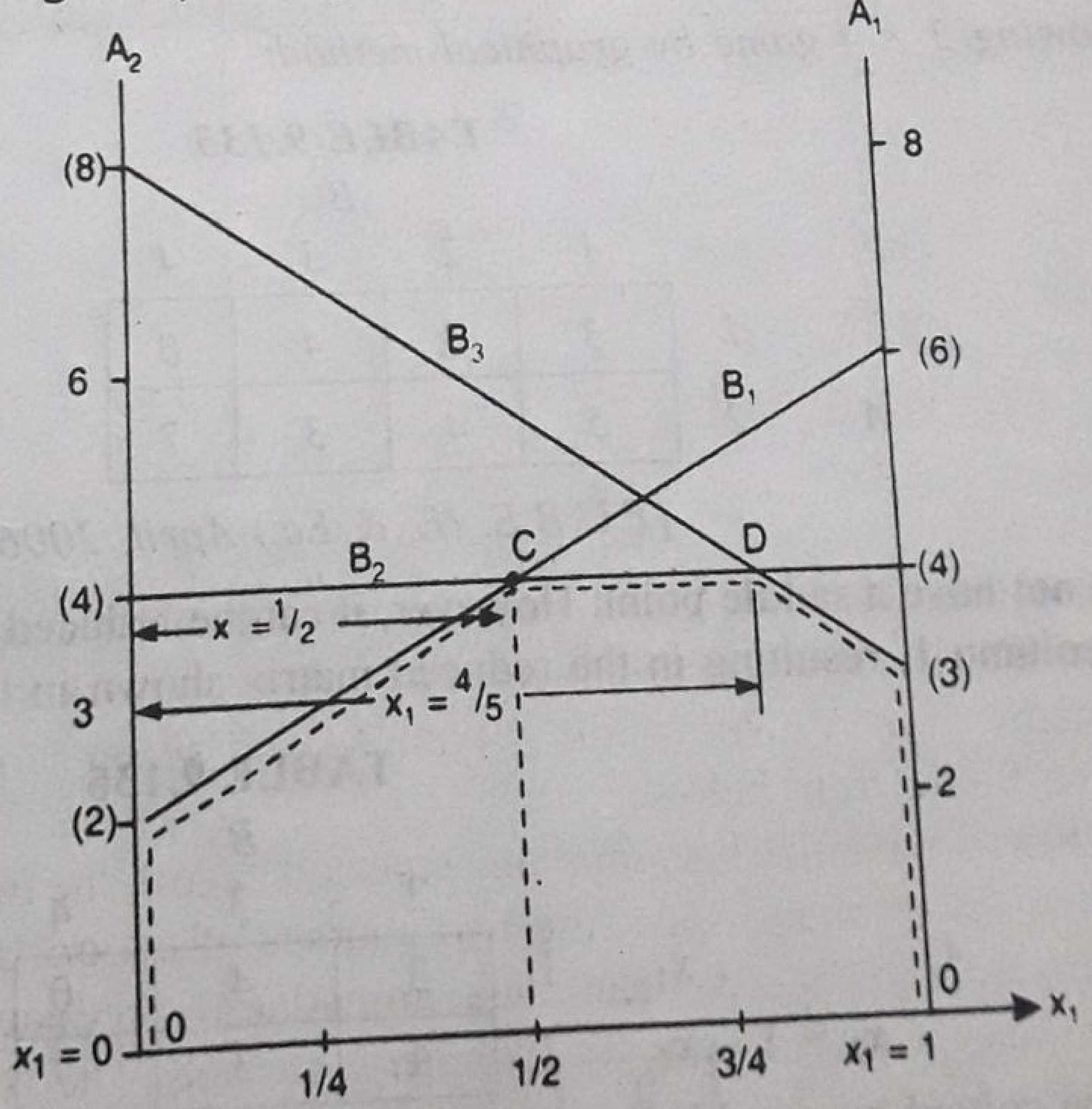



**Fig. 9.25**

| | | B | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | | |
| A | 1 | 6 | 4 | 2 | ½ |
| | 2 | 2 | 4 | 2 | ½ |
| | | 0 | 4 | | |
| | | 0 | 1 | | |

∴ Optimal solution is : A $\left(\frac{1}{2}, \frac{1}{2}\right)$, B (0, 1, 0) ; V = 4.

(*ii*) *Point D* : Strategies $B_2$ and $B_3$ are mixed by B and the following 2 × 2 game is to be solved:

| | | *B* 2 | 3 | | |
|---|---|---|---|---|---|
| A | 1 | 4 | 3 | 4 | $\frac{4}{5}$ |
| | 2 | 4 | 8 | 1 | $\frac{1}{5}$ |
| | | 5 | 0 | | |
| | | 1 | 0 | | |

∴ Optimal solution is A $\left(\frac{4}{5}, \frac{1}{5}\right)$, B (0, 1, 0) ; V = 4.

As evident from Fig. 9.25, any value of $x_1$ between the points C and D shall be optimal for A. Value of $x_1$ at points C and D is $\frac{1}{2}$ and $\frac{4}{5}$ respectively. Thus, optimal strategy for A is any pair of ($x_1$, $x_2 = 1 - x_1$), where $\frac{1}{2} \le x_1 \le \frac{4}{5}$ or $0.5 \le x_1 \le 0.8$. Optimal strategy for B is (0, 1, 0) and value of game is 4.

**Note:** *If there happen to be more than two lines passing through the maximin (or minimax) point, then any two lines having opposite signs for their slopes will yield an alternative optimal solution.*

**EXAMPLE 9.19-3.4**

*Solve the following 2 × 4 game by graphical method:*

*TABLE 9.135*

| | | *B* 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| | *1* | *3* | *3* | *4* | *0* |
| *A* | *2* | *5* | *4* | *3* | *7* |

**Solution** *[P.U.B.E. (E. & Ec.) April, 2006, B. Com. Sept., 2005]*

This game does not have a saddle point. However, it can be reduced by dominance because column 2 dominates column 1, resulting in the reduced matrix shown in table 9.136.

**TABLE 9.136**

| | | | *B* 2 | 3 | 4 |
|---|---|---|---|---|---|
| *A* | $x_1$ | 1 | 3 | 4 | 0 |
| | $x_2 = 1 - x_1$ | 2 | 4 | 3 | 7 |

This matrix can be solved by graphical method. A's expected payoffs corresponding to B's pure strategies are

| *B's pure strategies* | *A's expected payoffs* |
|---|---|
| 2 | $3x_1 + 4(1 - x_1) = -x_1 + 4$ |
| 3 | $4x_1 + 3(1 - x_1) = x_1 + 3$ |
| 4 | $0x_1 + 7(1 - x_1) = -7x_1 + 7$ |

These three straight lines can be plotted as functions of $x_1$. The method of plotting them has already been described; and they appear as shown in figure 9.26.

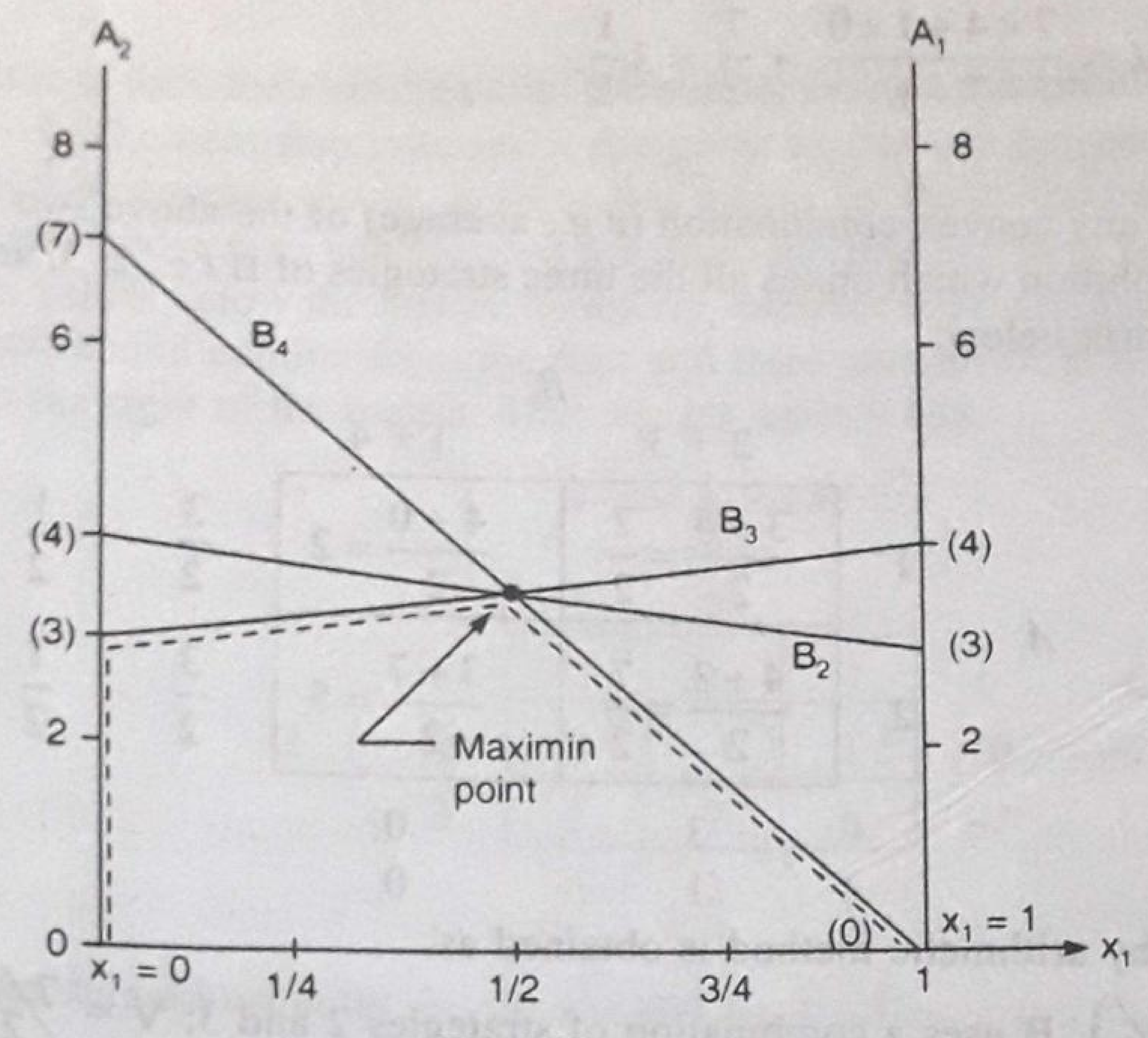



Fig. 9.26

Figure is to be bound from below as shown. All the three lines pass through the maximin point. As mentioned earlier, any two lines having opposite signs for their slopes will yield an alternative optimal solution. This means that the combination of $B_2$ and $B_4$ must be excluded as both have the same sign for their slopes. So the game reduces to two $2 \times 2$ games which can be easily solved, say, by arithmetic method.

(*i*)

| | | *B* 2 | 3 | | |
|---|---|---|---|---|---|
| *A* | 1 | 3 | 4 | 1 | 1/2 |
| | 2 | 4 | 3 | 1 | 1/2 |
| | | 1 | 1 | | |
| | | 1/2 | 1/2 | | |

A's optimum strategy : A (1/2, 1/2),
B's optimum strategy : B (0, 1/2, 1/2, 0),

$$\text{game value, } V = \frac{3 \times 1 + 4 \times 1}{1 + 1} = \frac{7}{2} = 3\frac{1}{2}.$$

(*ii*)

| | | *B* 3 | 4 | | | |
|---|---|---|---|---|---|---|
| *A* | 1 | 4 | 0 | 4 | 1/2 | 4/8 |
| | 2 | 3 | 7 | 4 | 1/2 | 4/8 |
| | | 7 | 1 | | | |
| | | 7/8 | 1/8 | | | |

A's optimum strategy : A (4/8, 4/8),
B's optimum strategy : B (0, 0, 7/8, 1/8),